# 2. Foundation of Probability Theory

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## 2.1 Random Experiments

**Definition (Random Experiment)**: A random experiment is a process that generates at least two possible outcomes. One and only one outcome will occur, but there is uncertainty associated with which one will occur.

Two essential elements of a random experiment:

* The set of all possible outcomes——**sample space**;
* The likelihood with which each outcome will occur——**probability function**.

Fundamental axioms of modern econometrics:

* An economic system can be viewed as a random experiment governed by some probability distribution or probability law.
* Any economic phenomena (often in form of data) can be viewed as an outcome of this random experiment.

## 2.2 Basic Concepts of Probability

**Definition (Sample Space)**: The possible outcomes of the random experiment are called **basic outcomes**, and the set of all basic outcomes is called the sample space, denoted by .

When an experiment is performed, the realization of the experiment is one outcome in the sample space.

**Example: Finite sample space**

|  |  |
| --- | --- |
| Tossing a coin |  |
| Rolling a die |  |
| Playing a football game |  |

**Example: Infinite discrete sample space**

Consider the number of accidents that occur at a given intersection within a month. The sample space is the set of all nonnegative integers,

**Example: Continuous sample space**

When recording the lifetime of a light bulb, the outcome is the time until the bulb burns out. Therefore the sample space is the set of all nonnegative real number, .

**Definition (Event)**: An event is a subset of basic outcomes from the sample space . The event is said to occur if the random experiment gives rise to one of the constituent basic outcomes in . That is, an event occurs if any of its basic outcomes has occurred.

**Example**

A die is rolled. The sample space Event is defined as "number resulting is even", then Event is defined as "number resulting is 4", then .

Remarks:

* The words "set" and "event" are interchangeable.
* Basic outcome sample space; event sample space.

## 2.3 Review of Set Theory

**Definition (Containment)**: The event is contained in the event , or contains if every sample point of is also a sample point of . If so, we write or equivalently, .

**Definition (Equality)**: Two events and are said to be equal, if and .

**Definition (Empty Set)**: The set containing no elements is called the empty set and is denoted by The event corresponding to is called a null (or impossible) event.

**Definition (Complement)**: The complement of denoted by is the set of basic outcomes of a random experiment belonging to but not to .

**Definition (Union)**: The union of and , denoted by is the set of all basic outcomes in that belong to either or . The union of and occurs if and only if either or (or both) occurs.

**Definition (Intersection)**: The intersection of and denoted by is the set of basic outcomes in that belong to both and . The intersection occurs if and only if both events and occur.

**Definition (Difference)**: The difference of and , denoted by or is the set of basic outcomes in that belong to but not to , i.e. .

**Definition (Exclusiveness)**: If and have no common basic outcomes, they are called mutually exclusive (or disjoint). Their intersection is empty set, i.e. .

**Definition (Collectively Exhaustiveness)**: Suppose are events in the sample space where is any positive integer. If then these events are said to be collectively exhaustive.

**Definition (Partition)**: A class of events forms a partition of the sample space if these events satisfy

1. for all (mutually exclusive), and
2. (collectively exhaustive).

Complementation

Commutativity

Associativity

Distributivity

More generally, for ,

De Morgan's Laws

More generally, for ,

**Definition (Countable Set)**: A set is called countable if the set has one-to-one correspondence with a subset of the natural numbers .

**Remarks:**

* A countable set is either a finite set or a countably infinite set.
* We can use a finite or infinite sequence to index all elements in a countable set.
* The set of natural numbers , the set of integers , and the set of rational numbers are countable sets.
* The set of real numbers is uncountable. The set of all real numbers in interval is uncountable.

Unions and intersections of infinitely countable collection of events:

Unions and intersections of (infinitely) uncountable collection of events: Let be an uncountable index set, then

## 2.4 Fundamental Probability Laws: -Algebra

**Definition -algebra**: A -algebra (or -field), denoted by is a collection of subsets of that satisfies

1. (the empty set is contained in );
2. If then is closed under complement); and
3. If then is closed under countable unions).

Remarks:

* A -algebra is a set of sets.
* (1) and (2) imply that .
* (2) and (3) imply that
* For a given we can construct many different -algebras.
* is a -algebra, usually called the trivial -algebra.
* The collection of all possible subsets of is a -algebra.
* For any event is a -algebra.
* Intersection of -algebras is also a -algebras.

**Theorem**: For any non-empty collection of subsets of sample space there exists a unique smallest -algebra containing . It is called the -algebra generated by .

* Smallest: If is a -algebra containing then
* Unique: If is another smallest -algebra containing then

**Example**: Suppose Let and .

* is a -algebra containing . (Smallest?)
* is the smallest -algebra containing

**Example (Borel algebra)**: Let the real line. Let be the collection of all sets that can be formed by taking complements, countable unions, and countable intersections of for any real numbers a and . Then is the smallest -algebra containing all the intervals. This -algebra is called Borel algebra, and any set in is called Borel set.

**Definition (Probability Function)**: Suppose a random experiment has a sample space and an associated -algebra . The probability function is a mapping that satisfies

1. for any event
2. If countable number of events are mutually exclusive (pairwise disjoint), then (countable additivity)

**Remark:** For a given measurable space many different probability functions can be defined.

**Definition (Probability Space)**: A probability space is a triple where

1. is the sample space corresponding to outcomes of the underlying random experiment;
2. is an associated -algebra of , and elements of are called events;
3. is a probability measure (or probability function).

Properties of probability function:

* If then
* If form a partition of (i.e. mutually exclusive and collectively exhaustive), and is an event in then
* Sub-additivity: For any sequence of events

**Theorem**: For any events

**Remark**: For

For the so-called **classical interpretation of probability**, we assume:

* Sample space contains a finite number of outcomes, say .
* All of the outcomes are equally probable.
* The associated -algebra is the collection of all possible subsets of

The probability of any single outcomes is For every event

How to determine the number of total outcomes in the sample space and in various events in ?

## 2.5 Methods of Counting

**Theorem (Fundamental Theorem of Counting)**: If a random experiment consists of separated tasks, the -th of which can be done in ways, then the entire job can be done in ways.

We consider two important counting methods: permutation and combination.

### 2.5.1 Permutation

**Example**: Suppose we will choose choose two letters out of in different orders, with each letter being used at most once each time. How many possible orders could we obtain?

12 ways:

Suppose there are boxes in a row and there are objects, where If we choose from the objects to fill in the boxes, how many possible different ordered sequences could we obtain?

1. One object is selected to fill in box so there are ways;
2. A second object is selected from the remaining objects to fill in box so there are ways;

* ... k. To fill the last box (box ), there are ways since objects remain.

The total number of different ways to fill boxes is

The experiment is equivalent to selecting objects out of the objects first, then arrange the selected objects in a sequence.

Each different arrangement of the sequence is called a **permutation**.

The number of permutations of choosing out of denoted by is

where .

Convention:

**Example (Birthday Problem)**: What is the probability that at least two persons in a group of people have the same birthday (i.e. the same day of the same month)? Assume no one is born on Feb 29 and each of the 365 days is equally likely to be the birthday of anyone.

Define event as "at least two persons have the same birthday".

* How many possible ways in which the people could be born?
* Event is " people have different birthdays". How many ways that all can have different birthdays?
* So

### 2.5.2 Combination

**Example**: Suppose we will choose two letters out of without ordering. If each letter is used at most once each time, how many possible pairs could we obtain?

6 pairs:

Choose a subset of elements without replacement from a set of distinct elements.

* The order of the elements is irrelevant. For example, the subsets and are identical.
* Each subset is called a **combination**.
* The number of combinations of choosing out of is denoted by We have

is also denoted by This is also called a binomial coefficient because of its appearance in the binomial theorem

Properties of the binomial coefficients

**Example**: Select 10 students randomly from a class containing 15 boys and 20 girls. What is the probability that exactly 3 boys are selected?

* The number of combinations of 10 students out of 35 students is
* The number of combinations of 3 boys out of 15 boys is
* The number of combinations of 7 girls out of 20 girls is

Thus the probability is .

**Matching Problem**: There are letters and envelopes with the corresponding addresses. If we place the letters in the envelopes in a random manner, what is the probability that at least one letter will be placed in the correct envelope?

Let be the event "at least one letter is in the correct envelope".

* Let be the event "letter is placed in the correct envelope". We shall determine
* Use formula
* For a given for any
* . Thus,

**Example**: Three types of fruit: apple, banana, and orange. How many ways to load a pack of 5 pieces?

Combination with replacement Number of ways to select subsets of size from a set of distinguishable elements:

|  |  |  |
| --- | --- | --- |
|  | Ordered | Unordered |
| Without replacement |  |  |
| With replacement |  |  |

## 2.6 Conditional Probability

Different economic events are generally related to each other. Because of the connection, the occurrence of event may affect or contain the information about the probability that event will occur.

**Example (Financial Contagion)**: A large drop of the price in one market can cause a large drop of the price in another market, given the speculations and reactions of market participants.

When an event has occurred, some uncertainty is eliminated as new information arrives. We want to update probability calculations based on new information.

**Definition (Conditional Probability)**: Let and be two events in probability space The conditional probability of event given event , denoted as is defined as

provided that .

Properties of conditional probability:

* Multiplication rule

**Example**: Suppose two balls are randomly selected, without replacement, from a box containing red balls and blue balls. What is the probability that the first is red and the second is blue?

Let and Then

**Theorem (Chain Rule of Joint Probability)**: For any events in the joint probability of events

is provided that

**Remark:** implies ,

**Theorem (Rule of Total Probability)**: Let be a **partition** (i.e. mutually exclusive and collectively exhaustive) of sample space with for all Then for any event in

**Example**: Suppose events and are mutually exclusive. If and for What is (Hint: and are also collectively exhaustive.)

## 2.7 Bayes' Formula

**Theorem (Bayes' Theorem)**: Suppose and Then

**Remarks:**

* We consider as the prior probability about event .
* is the posterior probability given that has occurred. ayes' Formula

**Theorem (Alternative Statement of Bayes' Theorem)**: Suppose are mutually exclusive and collectively exhaustive events in the sample space and is an event with Then the conditional probability of given is

**Example (Medical Test)**: Suppose there is a certain disease randomly found in 0.5% of the general population. Some blood test will yield a positive result in 99% of the case where the disease is present; but it also yields false-positive results in 5% of the case where a person is not infected. What is the probability that a person gets the disease if his test result is positive?

Let and denote the events "disease is present" and "positive test result", respectively. Then

**Example (Monty Hall Problem)**: In a TV game there are three doors and of which two hide nothing while behind the third there is a prize. The prize is won if it is guessed correctly that which door hides it.

* You choose one of the door first, say A.
* Before door is opened to reveal what is behind it, the game host open one of the other two doors, say and shows that there is nothing behind it.
* He then offers you the option to change your decision (from door to door ).

Should you stick to your original choice or change to

Let and be the events "prize is behind door A" (respectively, B and C). Assume Let be the event "host shows that nothing is behind door ". Then,

If the prize is behind door A, the hosts randomly open door B or door so

* If the prize is behind door B, then .
* If the prize is behind door , then

Therefore, and similarly, .

## 2.8 Independence

**Definition (Independence)**: Events and are said to be statistically independent if .

**Remarks:**

By definition,

Similarly, . Therefore, the occurrence of either one does not affect the probability of the occurrence of the other; that is, the knowledge of does not help in predicting .

Mutual exclusiveness does not necessarily imply independence.

If then any event is independent of .

**Theorem**: Let and be two independent events. Then (1) and and (3) and are all independent.

**Definition (Independence among Several Events)**: Events are (jointly) independent if, for every possible subset where

**Remark:**

There are conditions to be verified. For example, three events and are independent if

It is possible to find that three events are pairwise independent but not jointly independent.

**Example**: Suppose and each basic outcome is equally likely to occur. Let and Then,

but

It is also possible to find three events , that satisfy but not independent.

**Example**: Suppose and each basic outcome is equally likely to occur. Let and Then,

but

**Theorem**: If events are independent, then